



PERFORMANCE AND OPTIMIZATION CRITERIA FOR FORWARD AND REVERSE QUANTUM STIRLING CYCLES

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Abstract—The optimal performance for forward and reverse quantum Stirling cycles is studied in this paper. The finite time thermodynamic performance bound and optimization criteria for these cycles are derived. The optimal relations between the performance parameters for quantum Stirling engines, refrigerators and heat pumps are obtained. Published by Elsevier Science Ltd.

Harmonic oscillators
statistical mechanics

Quantum Stirling cycle

Finite-time thermodynamics

Nonequilibrium

INTRODUCTION

Stirling engines, refrigerators and heat pumps are thermally powered equipment with forward and reverse Stirling cycles. They have the advantages of high efficiency, low vibration levels, low noise, long service life, simple structure, etc; therefore, they have been used widely in power engineering, cryogenics and energy utility industries. Some authors have studied the performance optimization problem for the Stirling engine and refrigerator using the method of finite time thermodynamic analysis [1–4]. Significant results have been obtained by Ladas and Ibrahim [5], Wu [6], and Hu [7]. However, in the works mentioned above, the authors have all used classical thermodynamics to study the Stirling systems, while the quantum performance of the working fluid was not considered. Only recently, finite time thermodynamics has made considerable progress, and the object of study now has been extended from classical thermal systems to quantum thermal systems [8–13]. Geva and Kosloff [9, 10] proved that the optimal efficiency for a quantum Carnot engine at its maximum power output is the same as that derived by Curzon and Ahlborn [1]. Wu, Sun and Chen [14–16] obtained the optimal performance parameters for the quantum Carnot refrigerator and the quantum Stirling engine by solving the generalized master equation of an open system [17]. Based on Ref. [14] and using finite time thermodynamics and nonequilibrium quantum statistical theory, the performance optimizations for the quantum Stirling engine, refrigerator and heat pump are extended in this paper. The optimal performance bound and optimization criteria for the forward and reverse quantum Stirling cycles at several limiting cases are derived. The results provide a theoretical basis for optimal design and operation of the Stirling engine, refrigerator and heat pump. Notice that, in the calculation of this paper, we set $h' = h/2\pi = 1$, where h is Planck's constant.

QUANTUM STIRLING CYCLES

The quantum Stirling cycle is made of two isothermal branches and connected by two regenerating branches. The working fluid of the cycle consists of many noninteracting harmonic oscillators at the atomic level. The population of the oscillators (n) at thermal equilibrium can be obtained from the Bose–Einstein distribution [18]

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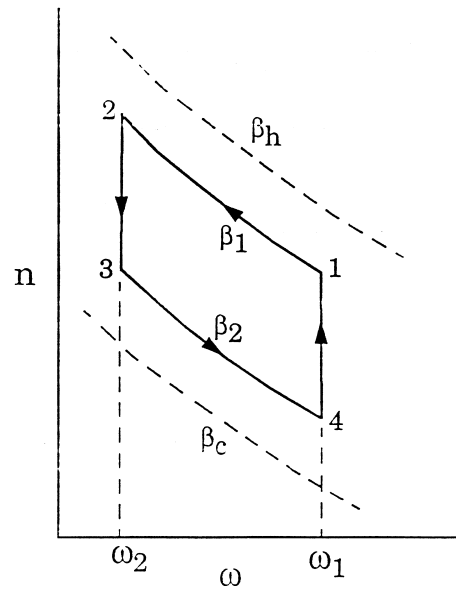


Fig. 1. Stirling forward cycle.

$$n = 1/[\exp(\beta\omega) - 1] \quad (1)$$

where $\omega > 0$ is the frequency of the oscillator, $\beta = 1/T$, and T is the internal temperature of the working fluid.

We assume that the cycle operates between an isothermal heat source B_1 at temperature T_h and an isothermal heat sink B_2 at temperature T_c . The heat source and sink are infinitely large and their internal relaxation is very strong, thus the disturbance of the working fluid on the source or the sink can be ignored. Therefore, the source B_1 and the sink B_2 are assumed to be in thermal equilibrium.

The internal temperatures of the working fluid are T_1 and T_2 in the two isothermal processes. We assume $T_2 > T_c$ (T_c is the critical temperature) such that the Bose–Einstein condensation of the working fluid will not be considered. The second law of thermodynamics requires $\beta_c > \beta_2 > \beta_1 > \beta_h$ for the forward cycle and $\beta_2 > \beta_c > \beta_h > \beta_1$ for the reverse cycle, where $\beta_i = 1/T_i$ ($i = h, c, 1$, and 2).

In the two regenerating processes, the volumes of the working fluid and the frequencies of the oscillator are both constants [14]. The frequencies of the oscillator in the two regenerations connecting the isotherms are ω_1 and ω_2 with $\omega_1 > \omega_2$. The forward and the reverse Stirling cycles are plotted in the (n, ω) plane as shown in Fig. 1 and Fig. 2, respectively.

Assuming that the internal temperature of the working fluid in the regenerating processes changes uniformly with respect to time [6], we have

$$dT/dt = d(\beta^{-1}/dt) = \pm 1/k, \quad k \neq 0 \quad (2)$$

where k is the time coefficient of the regenerations with constant frequencies. The signs “+” and “−” correspond to the processes 4–1 and 2–3, respectively.

Heat leak, regenerator loss and the other irreversibilities, except heat resistance, are not considered in this paper for simplicity. The cycle described above is the endoreversible quantum Stirling cycle.

THERMODYNAMIC ANALYSIS AND CYCLE PERIOD

The Hamiltonian of the harmonic oscillator system S , H_s , is described in the following form [6]

$$H_s = \omega N = \omega a^+ a \quad (3)$$

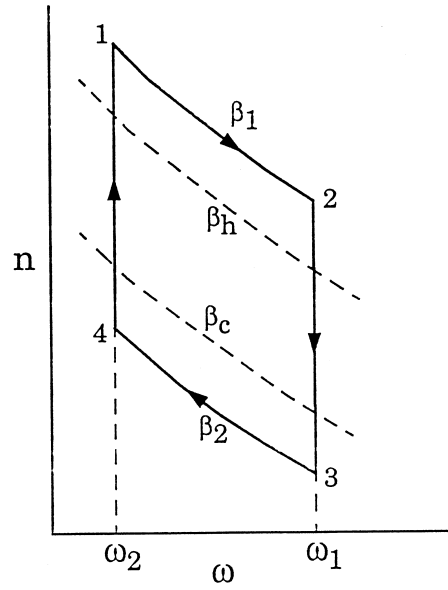


Fig. 2. Stirling reverse cycle.

where $\mathbf{N}$ is the number operator, $\mathbf{a}^+$ and $\mathbf{a}$ are the Bosonic creation and annihilation operators $\{\mathbf{N} = \mathbf{a}^+ \mathbf{a}, [\mathbf{a}, \mathbf{a}^+] = 1\}$.

From equation (3), the rate of change of energy of the working fluid is obtained by

$$dE/dt = d\langle \mathbf{H}_s \rangle / dt = (d\omega/dt)\mathbf{n} + \omega(d\mathbf{n}/dt). \quad (4)$$

Comparing equation (4) with the time derivative of the first law of thermodynamics, we obtain the instantaneous heat flow (Q) and the instantaneous power (P) in the following equations:

$$Q = \omega(d\mathbf{n}/dt) \quad (5)$$

$$P = - (d\omega/dt)\mathbf{n}. \quad (6)$$

Substituting equation (1) into equation (5) and expanding the heat quantity integral to first order in $\beta_i \omega$ ($i = h, 1$ or c , and 2) at the limit ($\mathbf{n} \gg 1$ and $\beta_i \omega < 1$) gives

$$Q_1 = m \int \omega d\mathbf{N} \lambda / \beta_1, \quad \mathbf{n}_1 < \mathbf{n} < \mathbf{n}_2 \quad (7)$$

$$Q_2 = \lambda / \beta_1. \quad (8)$$

where Q_1 and Q_2 are the heat exchange quantities in the isothermal processes 1–2 and 3–4, the constant $\lambda = \ln(\omega_1/\omega_2) > 0$ is the natural logarithm of the frequency ratio, and $m = \pm 1$ correspond to the forward and reverse cycles, respectively.

The heat exchanges of processes 2–3 and 4–1 are executed inside the engine, the refrigerator or the heat pump. If the regenerator is free of loss, the quantity of the regeneration is independent of the energy dissipation of the cycle. Thus, from equations (7) and (8), the work output from the forward cycle (or the work input required by the reverse cycle) is

$$W = Q_1 - Q_2 = \lambda(1/\beta_1 - 1/\beta_2). \quad (9)$$

In the two isothermal processes, the working fluid S becomes an open system owing to the coupling action between S and the heat source B_1 (or heat sink B_2). Using the generalized master equation of an open system, and in the Heisenberg picture, one obtains the motion equation of the operator $\mathbf{x}$ in the system S [14]

$$d\mathbf{x}/dt = i(\mathbf{L}_s + T_{\text{RB}} \mathbf{L}_{\text{SB}} \rho_{\text{B}}) \mathbf{x} - \sum \{(\omega_{kj})^+ [\mathbf{x}, \mathbf{Q}_k] \mathbf{Q}_j - (\omega_{kj})^- \mathbf{Q}_j [\mathbf{x}, \mathbf{Q}_k]\} \quad (10)$$

where $\mathbf{L}_s$ is the Liouville's operator of the system S , T_{RB} represents the trace of the operator of

the heat reservoir (B_1 or B_2), $\mathbf{L}_{sB}$ represents the coupling action between the working fluid S and the heat reservoir (B_1 or B_2), ρ_B is the density matrix of the heat reservoir (B_1 or B_2), $(\omega_{kj})^+$ and $(\omega_{jk})^-$ are constants related to the matrix elements of the density matrix ρ_B , $\mathbf{Q}_j$ and $\mathbf{Q}_k$ are the operators of the working fluid S and $[\cdot]$ is the quantum Poisson's bracket.

Setting $\mathbf{x} = \mathbf{N}$, $\mathbf{Q}_1 = \mathbf{a}^+$, and $\mathbf{Q}_2 = \mathbf{a}$, and neglecting the disturbance of the working fluid on the reservoir ($\mathbf{L}_s + \mathbf{T}_{rB}\mathbf{L}_{sB}\rho_B = 0$) gives

$$d\langle \mathbf{N} \rangle / dt = -(\alpha_1 - \alpha_2)n + \alpha_2 \quad (11a)$$

where $\alpha_1 = [(\omega_{12})^+ + (\omega_{12})^-]$ and $\alpha_2 = [(\omega_{21})^+ + (\omega_{21})^-]$ are both constants. The solution of equation (11a) is given by

$$n = n_e + (n_o - n_e) \exp[-(\alpha_1 - \alpha_2)t] \quad (12)$$

where n_e is the value of n at $t = 0$, while

$$n_e = \alpha_2 / (\alpha_1 - \alpha_2) \quad (13a)$$

n_e is the asymptotic stationary value of n . At thermal equilibrium, n_e must meet the following relation:

$$n_e = 1 / [\exp(\beta_j \omega) - 1]. \quad (13b)$$

Combining equation (13a) and (13b) gives $\alpha_1 = a \exp[(1 + q)\beta_j \omega]$ and $\alpha_2 = -a \exp(q\beta_j \omega)$.

Substituting α_1 and α_2 into equation (11a) yields

$$dn/dt = -a \exp(q\beta_j \omega) \{ [\exp(\beta_j \omega) - 1]n - 1 \} \quad (11a)$$

where a and q are both constants, $j = h$ and c correspond to the isothermal processes 1–2 and 3–4, respectively.

Substituting equation (1) into equation (11b), expanding the time integral to the second order in $\beta_j \omega$ at the limit ($n \gg 1$ and $\beta_i \omega < 1$) gives

$$t_1 = m \int [(dn/d\omega) / (dn/dt)] d\omega m A / (\beta_1 - \beta_h), \quad \omega_1 < \omega < \omega_2 \quad (14)$$

$$t_2 = m A / (\beta_e \beta_2) \quad (15)$$

where t_1 and t_2 are the times of the isotherms 1–2 and 3–4, A [$A = (1/\omega_1 - 1/\omega_2)/a > 0$] is a constant which is independent of temperature but dependent on the quantum performance of the working fluid, and $m = \pm 1$ corresponds to forward and reverse cycles, respectively.

From equation (2), the times t_3 and t_4 of the regeneration processes 2–3 and 4–1 are:

$$t_3 = t_4 = k(1/\beta_1 - 1/\beta_2) \quad (16)$$

The cycle period time, τ , is

$$\tau = t_1 + t_2 + t_3 + t_4. \quad (17a)$$

Substituting equations (14)–(16) into equation (17a) yields

$$\tau = A m [1/(\beta_1 - \beta_h) + 1/(\beta_e - \beta_2)] + 2k(1/\beta_1 - 1/\beta_2). \quad (17b)$$

PERFORMANCE BOUND AND OPTIMIZATION CRITERIA

Let us define the heat reservoir temperature ratio exponent r in the following form [19]

$$\beta_1/\beta_2 = (\beta_h/\beta_e)^r = D^r \quad (18)$$

where $D = \beta_h/\beta_e = T_c/T_h$ is the reservoir temperature ratio.

The power output (P), the efficiency (η) of the quantum Stirling engine, the cooling load (R), the COP (coefficient of performance, c) of the refrigerator, the heating load (H) and the COP (Ψ) of the heat pump can be obtained from equations (7)–(9) and (17b) and (18) in the

following equations:

$$P = W/\tau = (1 - D^r)f(r, \beta_2), \quad (19)$$

$$\eta = W/Q_1 = 1 - D^r, \quad 0 < r < 1 \quad (20)$$

$$R = Q_2/\tau = D^r f(r, \beta_2), \quad (21)$$

$$\epsilon = Q_2/W = D^r/(1 - D^r), \quad 1 < r < \infty \quad (22)$$

$$\Pi = Q_1/\tau = f(r, \beta_2), \quad (23)$$

$$\Psi = Q_1/W = 1/(1 - D^r), \quad 1 < r < \infty \quad (24)$$

and

$$f(r, \beta_2) = \lambda \{ Am \beta_2 D^r [1/(\beta_2 D^r - \beta_h) + 1/(\beta_e - \beta_2) + 2k(1 - D^r)]^{-1} \quad (25)$$

It is seen that, when r is constant, η , ϵ and Ψ are all constants. The maximum value $f_o(r)$ of $f(r, \beta_2)$ on β_2 can be obtained from equation (25).

$$f_o(r) = \lambda [Am D^{rmr} / (1 + D^{0.5})^2 / (D^r - D) + 2k(1 - D^r)]^{-1}. \quad (26)$$

When r is given, which means η , ϵ or Ψ is given, the optimal values of P , R and Π can be obtained from eqs (19), (21), (23) and (26) in the following equations, respectively:

$$P_o = (1 - D^r)f_o(r), \quad (27)$$

$$R_o = D^r f_o(r), \quad (28)$$

$$\Pi_o = f_o(r), \quad (29)$$

or

$$P_o = \lambda \eta [A(1 + D^{0.5})^2 (1 - \eta) / (1 - \eta - D) + 2k\eta]^{-1}, \quad (30)$$

$$R_o = \lambda [A(1 + D^{0.5})^2 (1 + \epsilon) / (D + \epsilon D - \epsilon) + 2k/\epsilon]^{-1}, \quad (31)$$

$$\Pi_o = \lambda [A(1 + D^{0.5})^2 (\Psi - 1) / (1 + \Psi D - \Psi) + 2k/\Psi]^{-1}. \quad (32)$$

Equations (30)–(32) are the main optimization relations between the power output and the efficiency of the engine, between the cooling load and the COP of the refrigerator, and between the heating load and the COP of the heat pump, respectively.

Using the extreme value theory of functions, the maximum values of power output, cooling load, and heating load can be obtained from equations (27)–(29) in the following equations:

$$P_m = \lambda \{ 2k + A[(1 + D^{0.5})(1 - D^{0.5})^2]^{-1} \}, \quad (33)$$

$$R_m = \lambda \{ [A^{0.5} + (A/D^{0.5}) + (2k/D)^{0.5}]^2 - 2k \}^{-1}, \quad (34)$$

$$\Pi_m = \lambda [2k(1 - D) - A(1 + D^{0.5})^2]^{-1}. \quad (35)$$

The extreme points corresponding to P_m , R_m , and Π_m are:

$$r_{m1} = 0.5, \quad (36)$$

$$r_{m1} = (\ln D)^{-1} \ln [D(2k)^{0.5} / (A^{0.5} + (2k)^{0.5} + (DA)^{0.5})], \quad (37)$$

$$r_{m1} = (\ln D)^{-1} \ln [D - (D + D^{0.5})(A/2k)^{0.5}]. \quad (38)$$

Substituting r_{m1} , r_{m2} and r_{m3} into equations (20), (22) and (24) gives

$$\eta_{m1} = 1 - D^{0.5} = \eta_{CA}, \quad (39)$$

$$\epsilon_m = D(2k)^{0.5} / [(A)^{0.5} + (2k)^{0.5} + (DA)^{0.5} - D(2Ak)^{0.25}], \quad (40)$$

$$\Psi_m = [1 - D + (D + D^{0.5})(A/2k)^{0.5}]^{-1}. \quad (41)$$

η_m , ϵ_m , and Ψ_m are finite time thermodynamic performance bounds for the quantum Stirling engine, refrigerator and heat pump, respectively.

For the forward cycle, when $r < r_{m1}$, P and η both decrease with the increase of r . When $r = r_{m1}$, $P = P_m$ and $\eta = \eta_m$. When $r_{m1} < r < 1$, P decreases but η increases with increase of r . When $r = 1$, $\eta = \eta_C$ ($\eta_C = 1 - D = 1 - \beta_h/\beta_c$ is the reversible Carnot cycle efficiency) and $P = 0$. Therefore, the finite time thermodynamic optimization criterion for the quantum Stirling engine should be in the following ranges:

$$r < r_{m1} < 1, \quad (42)$$

and

$$\eta_{CA} < \eta < \eta_C. \quad (43)$$

Using similar approaches as those for the heat engine, we obtain the optimization criteria of the quantum Stirling refrigerator and heat pump in the following equations:

$$1 < r < r_{m2}, \quad (44)$$

and

$$1 < r < r_{m3}, \quad (45)$$

respectively.

DISCUSSION

In equations (27)–(41), k represents the effect of the regenerating times (t_3 and t_4) on the optimum performance for the quantum Stirling cycles. Assuming the regenerating times in equations (25) and (26) are negligible ($k = 0$), we have

$$[f(r, \beta_2)]_{k=0} = \lambda \{ Am\beta_2 D^r [1/(\beta_2 D^r - \beta_h) + 1/(\beta_e - \beta_2)] \}^{-1}, \quad (46)$$

$$[f(r)]_{k=0} = \lambda (D^r - D) [AmD^r (1 + D^{0.5})^2]^{-1}. \quad (47)$$

Define the factors of the power (δ_1), cooling load (δ_2) and heating load (δ_3) of the forward and reverse quantum Stirling cycles as $\delta_1 = P/P_o$, $\delta_2 = R/R_o$, and $\delta_3 = \Pi/\Pi_o$. Let us also define a temperature coefficient (θ) as $\theta = [(\beta_2 D^r - \beta_h)/(\beta_2 D^r - D)]$. From equations (19), (21), (23) and (27)–(29) with $k = 0$, we have

$$\delta_1 = \delta_2 = \delta_3 = [f(r, \beta_2)/f_o(r)]_{k=0} = (1 + D^{0.5})^2 [1/\theta + D/(1 - \theta)]^{-1}. \quad (48)$$

Equation (48) indicates that the three factors are equal when $k = 0$. These factors have the maximum values $\delta_{1m} = \delta_{2m} = \delta_{3m} = 1$ when $\theta = \theta_o = 1/(1 + D^{0.5})$. Thus, when the regenerating time is neglected, the optimization criterion for the quantum Stirling cycles can be unified in the following form:

$$\theta = [(\beta_2 D^r - \beta_h)/(\beta_2 D^r - D)] = \theta_o = 1/(1 + D^{0.5}), \quad (49)$$

($0 < r < 1$ for forward cycle and $1 < r < \infty$ for reverse cycle).

CONCLUSION

Comparing equations (30)–(32) and (39)–(41) with the results in Refs [6, 7], we see that the quantum Stirling cycle is different from the classical thermodynamic Stirling cycle. The difference is because of the different character between the quantum working fluid and the classical thermodynamic working fluid. On one hand, the coupling action between the working fluid and the source B_1 (or the sink B_2) obeys the law of quantum statistical mechanics. On the other hand, the character of the quantum working fluid depends on the quantum parameters $\lambda = \ln(\omega_1/\omega_2)$ and $A = (1/\omega_1 - 1/\omega_2)/a$. For the Stirling cycles with quantum working fluids, such as Ne, He, D_2 etc., the quantum characters must be considered. The results in this paper

provide a sensitivity analysis for the optimum design and evaluation of the Stirling heat engine, refrigerator and heat pump.

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